

Algebra of formal power series, isomorphic to the algebra of formal Dirichlet series

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Abstract

Ordinary algebra of formal power series in one variable is convenient to study by means of the algebra of Riordan matrices and the Riordan group. In this paper we consider algebra of formal power series without constant term, isomorphic to the algebra of formal Dirichlet series. To study it, we introduce matrices, similar to the Riordan matrices. As a result, some analogies between two algebras becomes visible. For example, the Bell polynomials (polynomials of partitions of number n into m parts) play a certain role in the ordinary algebra. Similar polynomials (polynomials of decompositions of number n into m factors) play a similar role in the considered algebra. Analog of the Lagrange series for the considered algebra is also exists. In connection with this analogy, we introduce matrix group, similar to the Riordan group and called the Riordan-Dirichlet group. As an example, we consider analog of the Abel's identities for this group.

1 Introduction

Transformations, corresponding to multiplication and composition of series, play the main role in the space of formal power series over the field of real or complex numbers. Multiplication is given by the matrix $(a(x), x)$ n th column of which, $n = 0, 1, 2, \dots$, has the generating function $b(x)x^n$; composition is given by the matrix $(1, a(x))$ n th column of which has the generating function $a^n(x)$, $a_0 = 0$:

$$(b(x), x)g(x) = b(x)g(x), \quad (1, a(x))g(x) = g(a(x)).$$

Matrix

$$(b(x), x)(1, a(x)) = (b(x), a(x))$$

is called Riordan array [1] – [4]; n th column of Riordan array has the generating function $b(x)a^n(x)$. Thus,

$$(b(x), a(x))f(x)g^n(x) = b(x)f(a(x))(g(a(x)))^n,$$

$$(b(x), a(x))(f(x), g(x)) = (b(x)f(a(x)), g(a(x))).$$

Matrices $(b(x), a(x))$, $b_0 \neq 0$, $a_1 \neq 0$, form a group called the Riordan group.

n th coefficient of the series $a(x)$, n th row and n th column of the matrix A will be denoted respectively by

$$[x^n]a(x), \quad [n, \rightarrow]A, \quad [\uparrow, n]A,$$

at that $[x^n]a(x)b(x) = [x^n](a(x)b(x))$. We associate rows and columns of matrices with the generating functions of their elements.

Matrices

$$|e^x|^{-1}(b(x), a(x))|e^x| = (b(x), a(x))_{e^x},$$

where $|e^x|$ is the diagonal matrix whose diagonal elements are equal to the coefficients of the series e^x : $|e^x| a(x) = \sum_{n=0}^{\infty} a_n x^n / n!$, are called exponential Riordan arrays. Denote

$$[n, \rightarrow] (b(x), a(x))_{e^x} = s_n(x), \quad b_0 \neq 0, \quad a_1 \neq 0.$$

Then

$$(b(x), a(x))_{e^x} (1 - \varphi x)^{-1} = |e^x|^{-1} (b(x), a(x)) e^{\varphi x} = |e^x|^{-1} b(x) \exp(\varphi a(x)),$$

or

$$\sum_{n=0}^{\infty} \frac{s_n(\varphi)}{n!} x^n = b(x) \exp(\varphi a(x)).$$

Sequence of polynomials $s_n(x)$ is called Sheffer sequence, and in the case $b(x) = 1$, binomial sequence [4]. If $s_n(x)$ corresponds to the n th row of the matrix $(1, \log a(x))_{e^x}$, then

$$a^\varphi(x) = \sum_{n=0}^{\infty} \frac{s_n(\varphi)}{n!} x^n.$$

Matrices

$$(a(x), x) = \sum_{n=0}^{\infty} a_n(x^n, x),$$

$$(a(x), x) = \begin{pmatrix} a_0 & 0 & 0 & 0 & \dots \\ a_1 & a_0 & 0 & 0 & \dots \\ a_2 & a_1 & a_0 & 0 & \dots \\ a_3 & a_2 & a_1 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

form the algebra, isomorphic to the algebra of formal power series. Theme of this paper appeared as a result of the following observation. If in the algebra of matrices $(a(x), x) = \sum_{n=1}^{\infty} a_n(x^n, x)$, isomorphic to the algebra of formal power series without constant term, the matrix of multiplication (x^n, x) is replaced by the matrix of composition $(1, x^n)$, the result will be algebra, isomorphic to the algebra of formal Dirichlet series.

In the following sections of this paper we will consider the basic elementary aspects of the algebra thus obtained. Emphasis is on its relationship with ordinary algebra of formal power series. Research tools are the matrices, similar to the Riordan matrices. Group, similar to the Riordan group, is introduced in the last section, in which we consider series, similar to the Lagrange series

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Denote $(1, x^n) = \langle x^n, x \rangle$, where we take into consideration the analogy with Riordan matrices, which will be developed in the future. Then

$$\langle a(x), x \rangle = \sum_{n=1}^{\infty} a_n \langle x^n, x \rangle,$$

$$\langle a(x), x \rangle = \begin{pmatrix} a_\Sigma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_2 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_3 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_4 & a_2 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_5 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_6 & a_3 & a_2 & 0 & 0 & a_1 & 0 & 0 & 0 & \dots \\ 0 & a_7 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & \dots \\ 0 & a_8 & a_4 & 0 & a_2 & 0 & 0 & 0 & a_1 & 0 & \dots \\ 0 & a_9 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 & a_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$[\uparrow, n] \langle a(x), x \rangle = a(x^n), \quad a(1) = \sum_{n=1}^{\infty} a_n = a_\Sigma.$$

Sum of the coefficients of formal power series, obviously, in need of definition. Perhaps, each numerical series $\sum_{n=1}^{\infty} a_n$ has the value equal to a certain number or $\pm\infty$, which in the case of convergent series coincides with the its sum. For divergent numerical series, selection of the value corresponding to the sum of series, is ambiguous and depends on the accepted conditions [5]. We go around this problem and will consider the expression a_Σ as “formal numerical series”. Actions with the formal numerical series we define by the action with the corresponding power series: if $a(x) + b(x) = c(x)$, then $a_\Sigma + b_\Sigma = c_\Sigma$; if $\langle a(x), x \rangle b(x) = c(x)$, then $a_\Sigma b_\Sigma = c_\Sigma$. Numerical series, corresponding to the series $a(x^n)$, are considered to be identical; if $a(x) = \varphi x^n$, then $a_\Sigma = \varphi$.

Algebras of the matrices $(a(x), x)$, $\langle a(x), x \rangle$ and the corresponding algebras of formal power series we will be called the $(a(x), x)$ - algebra and the $\langle a(x), x \rangle$ - algebra. Denote

$$\langle a(x), x \rangle b(x) = a(x) \circ b(x), \quad b_0 = 0.$$

If $a(x) \circ b(x) = c(x)$, then $c_n = \sum_{d|n} a_d b_{n/d}$, where summation is over all divisors d of number n . Inverse to the series $a(x)$ we call the series $a^{(-1)}(x)$, which is defined by the identity

$$a(x) \circ a^{(-1)}(x) = a^{(0)}(x) = x.$$

This is consistent with the fact that $\langle x, x \rangle$ is the identity matrix: $x^n \circ a(x) = a(x^n)$. Denote also

$$a^{(n-1)}(x) \circ a(x) = a^{(n)}(x).$$

Note parallels between two algebras. Since

$$(x^n, x)(x^m, x) = (x^m, x)(x^n, x) = (x^{n+m}, x),$$

$$\langle x^n, x \rangle \langle x^m, x \rangle = \langle x^m, x \rangle \langle x^n, x \rangle = \langle x^{nm}, x \rangle,$$

then

$$(a(x), x)(b(x), x) = (b(x), x)(a(x), x);$$

$$\langle a(x), x \rangle \langle b(x), x \rangle = \langle b(x), x \rangle \langle a(x), x \rangle.$$

Since

$$(x^m, x)^n = (x^{mn}, x),$$

then the matrix $(a(x), x)$ is the power series:

$$(a(x), x) = \sum_{n=0}^{\infty} a_n(x, x)^n;$$

since

$$\langle x^m, x \rangle^n = \langle x^{m^n}, x \rangle,$$

then

$$\langle a(x), x \rangle = a_1 \langle x, x \rangle + \sum_{m=2}^{\infty} \sum_{n=1}^{\infty} a_{m^n} \langle x^m, x \rangle^n, \quad m \neq k^s, \quad s > 1.$$

Thus, matrices of the form

$$\langle a(x), x \rangle = \sum_{n=0}^{\infty} a_n \langle x^m, x \rangle^n, \quad m > 1,$$

being power series, form the algebra, isomorphic to the $(a(x), x)$ -algebra: if

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n,$$

then

$$\left(\sum_{n=0}^{\infty} a_n x^{m^n} \right) \circ \left(\sum_{n=0}^{\infty} b_n x^{m^n} \right) = \sum_{n=0}^{\infty} c_n x^{m^n}.$$

For example, for integers k ,

$$(x + x^m)^{(k)} = \sum_{n=0}^{\infty} \binom{k}{n} x^{m^n}.$$

In the $(a(x), x)$ -algebra the identity

$$\left(\sum_{n=0}^{\infty} a_n \beta^n x^n \right) \left(\sum_{n=0}^{\infty} b_n \beta^n x^n \right) = \sum_{n=0}^{\infty} c_n \beta^n x^n$$

holds for any values β . In the $\langle a(x), x \rangle$ -algebra the similar identity holds:

$$\left(\sum_{n=1}^{\infty} a_n n^\beta x^n \right) \circ \left(\sum_{n=1}^{\infty} b_n n^\beta x^n \right) = \sum_{n=1}^{\infty} c_n n^\beta x^n.$$

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We introduce matrices $\langle x | a(x) \rangle$, which will play the role of connecting link between the $(a(x), x)$, $\langle a(x), x \rangle$ -algebras:

$$[\uparrow, n] \langle x | a(x) \rangle = a^{(n)}(x), \quad a_1 = 0.$$

For example,

$$\left\langle x \left| \frac{x^2}{1-x} \right. \right\rangle = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 2 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 2 & 1 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 0 & 1 & 2 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 4 & 3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Denote

$$\langle x|a(x) \rangle f(x) = f \circ (a(x)), \quad f(x) = \sum_{n=0}^{\infty} f_n x^n;$$

$$\langle b(x), x \rangle \langle x|a(x) \rangle = \langle b(x)|a(x) \rangle.$$

Product of matrices of the form $\langle b(x)|a(x) \rangle$ is not a matrix of the same form, but since

$$\langle x|a(x) \rangle x^m f(x) = a^{(m)}(x) \circ f \circ (a(x)),$$

then

$$\begin{aligned} \langle x|a(x) \rangle (f(x), x) &= \langle f \circ (a(x))|a(x) \rangle, \\ \langle x|a(x) \rangle f(x) c(x) &= f \circ (a(x)) \circ c \circ (a(x)), \\ \langle x|a(x) \rangle (1, g(x)) &= \langle x|g \circ (a(x)) \rangle, \\ \langle b(x)|a(x) \rangle (f(x), g(x)) &= \langle b(x) \circ f \circ (a(x))|g \circ (a(x)) \rangle. \end{aligned} \quad (1)$$

Thus, any matrix of the form $\langle b(x)|a(x) \rangle$ can be represented as the product of matrix of the same form and Riordan array.

Here we get the definitions of the power and of the logarithm for the $\langle a(x), x \rangle$ -algebra. Denote

$$a^{(\varphi)}(x) = \langle x|a(x) - x \rangle (1+x)^\varphi, \quad a_1 = 1.$$

Then

$$a^{(\varphi)}(x) \circ a^{(\beta)}(x) = a^{(\varphi+\beta)}(x), \quad (a^{(\varphi)}(x))^{(\beta)} = a^{(\varphi\beta)}(x).$$

Denote

$$\log \circ a(x) = \langle x|a(x) - x \rangle \log(1+x), \quad a_1 = 1.$$

Then

$$\begin{aligned} \log \circ a^{(\varphi)}(x) &= \varphi \log \circ a(x), \quad \log \circ (a(x) \circ b(x)) = \log \circ a(x) + \log \circ b(x), \\ \langle x|\log \circ a(x) \rangle e^{\varphi x} &= a^{(\varphi)}(x), \quad a^{(\varphi)}(x) \circ b^{(\varphi)}(x) = (a(x) \circ b(x))^{(\varphi)}. \end{aligned}$$

Denote

$$|e^x|^{-1} \langle x|\log \circ a(x) \rangle |e^x| = \langle x|\log \circ a(x) \rangle_{e^x}, \quad [n, \rightarrow] \langle x|\log \circ a(x) \rangle_{e^x} = \tilde{s}_n(x).$$

Then

$$a^{(\varphi)}(x) = \sum_{n=0}^{\infty} \frac{\tilde{s}_n(\varphi)}{n!} x^n.$$

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We note the following analogy between the matrices $(1, a(x))$ and $\langle x|a(x) \rangle$. Denote

$$B_{n,m}(a_1, a_2, \dots, a_n) = \sum \frac{m!}{m_1! m_2! \dots m_n!} a_1^{m_1} a_2^{m_2} \dots a_n^{m_n}, \quad n > 0,$$

where expression $\prod_{k=1}^n a_k^{m_k}$ corresponding to the partition $n = \sum_{k=1}^n k m_k$, $\sum_{k=1}^n m_k = m$ and summation is done over all partitions of number n into m parts. Since

$$\begin{aligned} \left(\sum_{k=1}^n a_k x^k \right)^m &= \sum_{m_1+m_2+\dots+m_n=m} \frac{m!}{m_1! m_2! \dots m_n!} (a_1 x)^{m_1} (a_2 x^2)^{m_2} \dots (a_n x^n)^{m_n} = \\ &= \sum_{m_1+m_2+\dots+m_n=m} \frac{m!}{m_1! m_2! \dots m_n!} a_1^{m_1} a_2^{m_2} \dots a_n^{m_n} x^\Sigma, \quad \Sigma = m_1 + 2m_2 + \dots + nm_n, \end{aligned}$$

then

$$[x^n] \left(\sum_{n=1}^{\infty} a_n x^n \right)^m = B_{n,m}(a_1, a_2, \dots, a_n), \quad [n, \rightarrow](1, a(x)) = \sum_{m=1}^n B_{n,m}(a_1, a_2, \dots, a_n) x^m :$$

$$(1, a(x)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_1 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_2 & a_1^2 & 0 & 0 & 0 & \dots \\ 0 & a_3 & 2a_1a_2 & a_1^3 & 0 & 0 & \dots \\ 0 & a_4 & 2a_1a_3 + a_2^2 & 3a_1^2a_2 & a_1^4 & 0 & \dots \\ 0 & a_5 & 2a_1a_4 + 2a_2a_3 & 3a_1^2a_3 + 3a_1a_2^2 & 4a_1^3a_2 & a_1^5 & \dots \\ 0 & a_6 & 2a_1a_5 + 2a_2a_4 + a_3^2 & 3a_1^2a_4 + 6a_1a_2a_3 + a_2^3 & 4a_1^3a_3 + 6a_1^2a_2^2 & 5a_1^4a_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

If $a_0 = 1$, $\log a(x) = b(x)$, $s_n(x) = [n, \rightarrow](1, b(x))_{ex}$, then

$$s_n(x) = n! \sum_{m=1}^n \frac{B_{n,m}(b_1, b_2, \dots, b_n)}{m!} x^m, \quad b_n = \sum_{m=1}^n (-1)^{m+1} \frac{B_{n,m}(a_1, a_2, \dots, a_n)}{m}.$$

Denote

$$\tilde{B}_{n,m}(a_2, a_3, \dots, a_n) = \sum \frac{m!}{m_2!m_3! \dots m_n!} a_2^{m_2} a_3^{m_3} \dots a_n^{m_n}, \quad n > 1,$$

where expression $\prod_{k=2}^n a_k^{m_k}$ corresponding to the decomposition $n = \prod_{k=2}^n k^{m_k}$, $\sum_{k=2}^n m_k = m$, and summation is done over all decompositions of number n into m factors. Since

$$\begin{aligned} & \left(\sum_{k=2}^n a_k x^k \right)^{(m)} = \\ &= \sum_{m_2+m_3+\dots+m_n=m} \frac{m!}{m_2!m_3! \dots m_n!} (a_2 x^2)^{(m_2)} \circ (a_3 x^3)^{(m_3)} \circ \dots \circ (a_n x^n)^{(m_n)} = \\ &= \sum_{m_2+m_3+\dots+m_n=m} \frac{m!}{m_2!m_3! \dots m_n!} a_2^{m_2} a_3^{m_3} \dots a_n^{m_n} x^\Pi, \quad \Pi = 2^{m_2} 3^{m_3} \dots n^{m_n}, \end{aligned}$$

then

$$[x^n] \left(\sum_{n=2}^{\infty} a_n x^n \right)^{(m)} = \tilde{B}_{n,m}(a_2, a_3, \dots, a_n), \quad [n, \rightarrow] \langle x | a(x) \rangle = \sum_{m=1}^n \tilde{B}_{n,m}(a_2, a_3, \dots, a_n) x^m :$$

$$\langle x|a(x)\rangle = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_2 & 0 & 0 & 0 & \dots \\ 0 & a_3 & 0 & 0 & 0 & \dots \\ 0 & a_4 & a_2^2 & 0 & 0 & \dots \\ 0 & a_5 & 0 & 0 & 0 & \dots \\ 0 & a_6 & 2a_2a_3 & 0 & 0 & \dots \\ 0 & a_7 & 0 & 0 & 0 & \dots \\ 0 & a_8 & 2a_2a_4 & a_2^3 & 0 & \dots \\ 0 & a_9 & a_3^2 & 0 & 0 & \dots \\ 0 & a_{10} & 2a_2a_5 & 0 & 0 & \dots \\ 0 & a_{11} & 0 & 0 & 0 & \dots \\ 0 & a_{12} & 2a_2a_6 + 2a_4a_3 & 3a_2^2a_3 & 0 & \dots \\ 0 & a_{13} & 0 & 0 & 0 & \dots \\ 0 & a_{14} & 2a_2a_7 & 0 & 0 & \dots \\ 0 & a_{15} & 2a_3a_5 & 0 & 0 & \dots \\ 0 & a_{16} & 2a_2a_8 + a_4^2 & 3a_2^2a_4 & a_2^4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

If $a_1 = 1$, $\log \circ a(x) = b(x)$, $\tilde{s}_n(x) = [n, \rightarrow] \langle x|b(x)\rangle_{ex}$, then

$$\tilde{s}_n(x) = n! \sum_{m=1}^n \frac{\tilde{B}_{n,m}(b_2, b_3, \dots, b_n)}{m!} x^m, \quad b_n = \sum_{m=1}^n (-1)^{m+1} \frac{\tilde{B}_{n,m}(a_2, a_3, \dots, a_n)}{m}.$$

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Relationship between the $(a(x), x)$, $\langle a(x), x \rangle$ -algebras is most visibly manifested in the following theorem.

Theorem 1. *Each formal power series $a(x)$, $a_0 = 1$,*

$$a^\varphi(x) = \sum_{n=0}^{\infty} \frac{s_n(\varphi)}{n!} x^n, \quad s_n(x) = [n, \rightarrow] (1, \log a(x))_{ex},$$

corresponds to the series $\tilde{a}(x)$,

$$\tilde{a}^{(\varphi)}(x) = x + \sum_{n=2}^{\infty} \frac{s_{m_1}(\varphi) s_{m_2}(\varphi) \dots s_{m_r}(\varphi)}{m_1! m_2! \dots m_r!} x^n, \quad n = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r},$$

where $n = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r}$ is the canonical decomposition of number n .

Proof. We will denote a prime number of the letter p . Consider series

$$\tilde{a}^{(\varphi)}(x) = \prod_{p=2}^{\infty} \circ \left(\sum_{n=0}^{\infty} a_{p,n} x^{p^n} \right)^{(\varphi)} = \prod_{p=2}^{\infty} \circ \tilde{a}_p^{(\varphi)}(x), \quad a_{p,0} = 1,$$

where product, denoted similar to the ordinary product, is taken over all prime numbers. In view of the isomorphism between the algebra of matrices $\langle a(x), x \rangle = \sum_{n=0}^{\infty} a_n \langle x^n, x \rangle^n$ and the algebra of matrices $(a(x), x)$, the series $\tilde{a}_p(x)$ corresponds to the series $a_p(x)$, such that

$$[x^{p^n}] \tilde{a}_p^{(\varphi)}(x) = [x^n] a_p^\varphi(x) = \frac{s_{p,n}(\varphi)}{n!}, \quad s_{p,n}(x) = [n, \rightarrow] (1, \log a_p(x))_{ex}.$$

Hence,

$$\tilde{a}^{(\varphi)}(x) = x + \sum_{n=2}^{\infty} \frac{s_{p_1, m_1}(\varphi) s_{p_2, m_2}(\varphi) \dots s_{p_r, m_r}(\varphi)}{m_1! m_2! \dots m_r!} x^n, \quad n = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r}.$$

We are interested in the case when all the series $\tilde{a}_p(x)$ corresponds to the same series $a(x)$. In this case the series $\tilde{a}(x)$ form a group, isomorphic to a group of the series $a(x)$: if $a(x)b(x) = c(x)$, then $\tilde{a}(x) \circ \tilde{b}(x) = \tilde{c}(x)$. Note that

$$[x^n] \log \circ \tilde{a}(x) = 0, \quad n \neq p^m; \quad = [x^m] \log a(x), \quad n = p^m.$$

Denote $\zeta(x) = \sum_{n=1}^{\infty} x^n$. Then

$$\zeta^{(\varphi)}(x) = x + \sum_{n=2}^{\infty} \frac{(\varphi)^{m_1} (\varphi)^{m_2} \dots (\varphi)^{m_r}}{m_1! m_2! \dots m_r!} x^n, \quad n = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r},$$

where $(\varphi)^{m_i} = \varphi(\varphi+1)(\varphi+2) \dots (\varphi+m_i-1)$. Denote

$$[n, \rightarrow] \langle x | \log \circ \zeta(x) \rangle_{e^x} = \tilde{s}_n(x).$$

Then

$$\tilde{s}_0(x) = 0, \quad \tilde{s}_1(x) = 1, \quad \frac{\tilde{s}_n(x)}{n!} = \frac{(x)^{m_1} (x)^{m_2} \dots (x)^{m_r}}{m_1! m_2! \dots m_r!}.$$

A-priori,

$$[x^n] \log \circ \zeta(x) = 0, \quad n \neq p^m; \quad = \frac{1}{m}, \quad n = p^m.$$

On the other hand,

$$[x^n] \log \circ \zeta(x) = \sum_{m=1}^n (-1)^{m+1} \frac{\tilde{B}_{n,m}(1, 1, \dots, 1)}{m},$$

where

$$\tilde{B}_{n,m}(1, 1, \dots, 1) = \sum \frac{m!}{m_2! m_3! \dots m_n!}, \quad n = \prod_{k=2}^n k^{m_k}, \quad \sum_{k=2}^n m_k = m,$$

and summation is done over all decompositions of number n into m factors. We note also the identity

$$\sum_{m=1}^n \binom{\varphi}{m} \tilde{B}_{n,m}(1, 1, \dots, 1) = \binom{\varphi + s_1 - 1}{s_1} \binom{\varphi + s_2 - 1}{s_2} \dots \binom{\varphi + s_r - 1}{s_r},$$

$n = p_1^{s_1} p_2^{s_2} \dots p_r^{s_r}$, similar to the identity

$$\sum_{m=1}^n \binom{\varphi}{m} B_{n,m}(1, 1, \dots, 1) = \binom{\varphi + n - 1}{n},$$

where

$$B_{n,m}(1, 1, \dots, 1) = \binom{n-1}{m-1} = \sum \frac{m!}{m_1! m_2! \dots m_n!}, \quad n = \sum_{k=1}^n k m_k, \quad \sum_{k=1}^n m_k = m,$$

and summation is done over all partitions of number n into m parts.

Analog of the exponential series for the $\langle a(x), x \rangle$ -algebra is the series

$$\varepsilon^{(\varphi)}(x) = x + \sum_{n=2}^{\infty} \frac{\varphi^{m_1+m_2+\dots+m_r}}{m_1! m_2! \dots m_r!} x^n, \quad n = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r}.$$

Series $\log \circ \varepsilon(x)$ is closely connected with the sequence of prime numbers:

$$[x^n] \log \circ \varepsilon(x) = 0, \quad n \neq p; \quad = 1, \quad n = p.$$

In general case

$$(\log \circ \varepsilon(x))^{(m)} = \sum \frac{m!}{m_1! m_2! \dots m_r!} x^n,$$

where summation is done over all $n = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r}$, $m_1 + m_2 + \dots + m_r = m$.

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It follows from the Lagrange series expansion for arbitrary formal power series $b(x)$ and $a(x)$, $a_0 = 1$:

$$\frac{b(x)}{1 - x(\log a(x))'} = \sum_{n=0}^{\infty} \frac{x^n}{a^n(x)} [x^n] b(x) a^n(x)$$

that each formal power series $a(x)$, $a_0 = 1$, is associated by means of the transform

$$a^\varphi(x) = \sum_{n=0}^{\infty} \frac{x^n}{a^{\beta n}(x)} [x^n] (1 - x\beta(\log a(x))') a^{\varphi+\beta n}(x)$$

with the set of series ${}_{(\beta)}a(x)$, ${}_{(0)}a(x) = a(x)$, such that

$$\begin{aligned} {}_{(\beta)}a(xa^{-\beta}(x)) &= a(x), & a(x{}_{(\beta)}a^\beta(x)) &= {}_{(\beta)}a(x), \\ [x^n] {}_{(\beta)}a^\varphi(x) &= [x^n] \left(1 - x\beta \frac{a'(x)}{a(x)}\right) a^{\varphi+\beta n}(x) = \frac{\varphi}{\varphi + \beta n} [x^n] a^{\varphi+\beta n}(x), \\ [x^n] \left(1 + x\beta \frac{{}_{(\beta)}a'(x)}{{}_{(\beta)}a(x)}\right) {}_{(\beta)}a^\varphi(x) &= \frac{\varphi + \beta n}{\varphi} [x^n] {}_{(\beta)}a^\varphi(x) = [x^n] a^{\varphi+\beta n}(x). \\ (1, x{}_{(\beta)}a^\beta(x))^{-1} &= (1, xa^{-\beta}(x)), \\ \left(1 + x\beta(\log {}_{(\beta)}a(x))', x{}_{(\beta)}a^\beta(x)\right)^{-1} &= (1 - x\beta(\log a(x))', xa^{-\beta}(x)), \\ [n, \rightarrow] (1, x{}_{(\beta)}a^\beta(x)) &= [n, \rightarrow] (1 - x\beta(\log a(x))' a^{\beta n}(x), x), \\ [n, \rightarrow] \left(1 + x\beta(\log {}_{(\beta)}a(x))', x{}_{(\beta)}a^\beta(x)\right) &= [n, \rightarrow] (a^{\beta n}(x), x). \end{aligned}$$

Denote

$$[n, \rightarrow] (1, \log {}_{(\beta)}a(x))_{e^x} = {}_{(\beta)}s_n(x), \quad {}_{(0)}s_n(x) = s_n(x).$$

Then

$${}_{(\beta)}a^\varphi(x) = \sum_{n=0}^{\infty} \frac{\varphi}{\varphi + \beta n} \frac{s_n(\varphi + \beta n)}{n!} x^n, \quad {}_{(\beta)}s_n(x) = x(x + \beta n)^{-1} s_n(x + \beta n).$$

Apparently, the series ${}_{(\beta)}a(x)$ for integer β , denoted by $S_\beta(x)$, were first considered in [6]. In [7] these series, called generalized Lagrange series, are considered in connection with the Riordan arrays. Examples of this construction are the generalized binomial and generalized exponential series [8; p. 200]:

$$a(x) = 1 + x, \quad {}_{(\beta)}a^\varphi(x) = \sum_{n=0}^{\infty} \frac{\varphi}{\varphi + \beta n} \binom{\varphi + \beta n}{n} x^n;$$

$$a(x) = e^x, \quad {}_{(\beta)}a^\varphi(x) = \sum_{n=0}^{\infty} \frac{\varphi(\varphi + \beta n)^{n-1}}{n!} x^n.$$

We introduce analog of the differential operator for the $\langle a(x), x \rangle$ -algebra:

$$\tilde{D}a(x) = a^*(x) = \sum_{n=1}^{\infty} \ln n a_n x^n.$$

Since

$$\ln n \sum_{d|n} a_d b_{n/d} = \sum_{d|n} \ln(n/d) a_d b_{n/d} + \sum_{d|n} \ln d a_d b_{n/d},$$

then

$$(a(x) \circ b(x))^* = a(x) \circ b^*(x) + a^*(x) \circ b(x), \quad (a^{(n)}(x))^* = na^{(n-1)}(x) \circ a^*(x),$$

$$(\log \circ a(x))^* = a^*(x) \circ \sum_{n=1}^{\infty} (-1)^{n-1} (a(x) - x)^{(n-1)} = a^*(x) \circ a^{(-1)}(x).$$

Since

$$a^{(\varphi)}(x) = \langle x | a(x) - x \rangle (1+x)^\varphi, \quad \tilde{D} \langle x | a(x) - x \rangle = \langle a^*(x) | a(x) - x \rangle D,$$

where D is the matrix of differential operator, then

$$(a^{(\varphi)}(x))^* = \varphi a^{(\varphi-1)}(x) \circ a^*(x).$$

Theorem 2. Each formal power series $a(x)$, $a_0 = 0$, $a_1 = 1$, is associated by means of the transform

$$a^{(\varphi)}(x) = \sum_{n=1}^{\infty} a^{(-\beta \ln n)}(x^n) [x^n] (x - \beta(\log \circ a(x))^*) \circ a^{(\varphi + \beta \ln n)}(x)$$

with the set of series ${}_{(\beta)}a(x)$, ${}_{(0)}a(x) = a(x)$, such that

$$\begin{aligned} [x^n] {}_{(\beta)}a^{(\varphi)}(x) &= [x^n] (x - \beta a^*(x) \circ a^{(-1)}(x)) \circ a^{(\varphi + \beta \ln n)}(x) = \\ &= \frac{\varphi}{\varphi + \beta \ln n} [x^n] a^{(\varphi + \beta \ln n)}(x), \\ [x^n] (x + \beta {}_{(\beta)}a^*(x) \circ {}_{(\beta)}a^{(-1)}(x)) \circ {}_{(\beta)}a^{(\varphi)}(x) &= \frac{\varphi + \beta \ln n}{\varphi} [x^n] {}_{(\beta)}a^{(\varphi)}(x) = \\ &= [x^n] a^{(\varphi + \beta \ln n)}(x). \end{aligned}$$

For proof we introduce the matrices $\langle x, a(x) \rangle$:

$$\langle x, a(x) \rangle = \begin{pmatrix} (a_\Sigma)^{\ln 0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & a_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & a_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & a_2^2 & 0 & a_1^4 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & a_1^5 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & a_3^2 & a_2^3 & 0 & 0 & a_1^6 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1^7 & 0 & 0 & \dots \\ 0 & 0 & a_4^2 & 0 & a_2^4 & 0 & 0 & 0 & a_1^8 & 0 & \dots \\ 0 & 0 & 0 & a_3^3 & 0 & 0 & 0 & 0 & 0 & a_1^9 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$a_n^m = [x^n] a^{(\ln m)}(x), \quad [\uparrow, n] \langle x, a(x) \rangle = a^{(\ln n)}(x^n).$$

Denote

$$\langle x, a(x) \rangle b(x) = b_d \circ (a(x)), \quad [x^n] b_d \circ (a(x)) = \sum_{d|n} b_d a_{n/d}^d;$$

$$\langle b(x), x \rangle \langle x, a(x) \rangle = \langle b(x), a(x) \rangle.$$

Since

$$a^{(\ln n)}(x^n) = x^n \circ a^{(\ln n)}(x), \quad n > 0,$$

Then

$$[\uparrow, n] \langle b(x), a(x) \rangle = x^n \circ b(x) \circ a^{(\ln n)}(x).$$

Let $b_d \circ (a(x)) = c(x)$. If we accept the rules

$$(a_\Sigma)^{\ln 0} b_\Sigma = c_\Sigma (a_\Sigma)^{\ln 0}, \quad (a_\Sigma)^{\ln 0} (b_\Sigma)^{\ln 0} = (a_\Sigma c_\Sigma)^{\ln 0}, \quad (1)^{\ln 0} = 1,$$

then following theorem is true.

Theorem 3. *Matrices $\langle b(x), a(x) \rangle$, $b_1 \neq 0$, $a_1 \neq 0$, form a group whose elements are multiplied by the rule*

$$\langle b(x), a(x) \rangle \langle f(x), g(x) \rangle = \langle b(x) \circ f_d \circ (a(x)), a(x) \circ g_d \circ (a(x)) \rangle.$$

Proof. Since

$$\langle x, a(x) \rangle b(x^m) = \sum_{n=1}^{\infty} b_n a^{(\ln mn)}(x^{mn}) = x^m \circ a^{(\ln m)}(x) \circ \sum_{n=1}^{\infty} b_n a^{(\ln n)}(x^n),$$

then

$$\langle x, a(x) \rangle \langle b(x), x \rangle = \langle b_d \circ (a(x)), a(x) \rangle.$$

Thus,

$$\langle x, a(x) \rangle b(x) \circ c(x) = b_d \circ (a(x)) \circ c_d \circ (a(x)).$$

Since

$$\langle x, a(x) \rangle b^{(\ln m)}(x^m) = x^m \circ a^{(\ln m)}(x) \circ (b_d \circ (a(x)))^{(\ln m)},$$

then

$$\langle x, a(x) \rangle \langle x, b(x) \rangle = \langle x, a(x) \circ b_d \circ (a(x)) \rangle.$$

As we shall see, with respect to the some structure that in the ordinary algebra of formal power series corresponds to the Lagrange series, a complete analogy exists between the group of matrices $\langle b(x), a(x) \rangle$, $b_1 \neq 0$, $a_1 \neq 0$, and the Riordan group. So we call this group the Riordan-Dirichlet group.

Note the identity for the matrices $\langle b(x) | a(x) \rangle$, complementary to identity (1):

$$\langle f(x), g(x) \rangle \langle b(x) | a(x) \rangle = \langle f(x) \circ b_d \circ (g(x)) | a_d \circ (g(x)) \rangle.$$

Now we prove the theorem 2. Let the matrices $\langle x, a^{(-1)}(x) \rangle$, $\langle x, b(x) \rangle$ are mutually inverse. Then

$$\langle x, a^{(-1)}(x) \rangle b(x) = a(x), \quad \langle x, b(x) \rangle a(x) = b(x).$$

Let $\tilde{D}1 = 0$ (perhaps we should accept $\tilde{D}1 = \ln 0$, but this is not fundamentally now). Since

$$\tilde{D}b^{(\ln n)}(x^n) = (x^n \circ b^{(\ln n)}(x))^* = \ln n x^n \circ b^{(\ln n)}(x) \circ (x + b^*(x) \circ b^{(-1)}(x)),$$

then

$$\tilde{D} \langle x, b(x) \rangle = \langle x + (\log \circ b(x))^*, b(x) \rangle \tilde{D}, \quad \langle x + (\log \circ b(x))^*, b(x) \rangle a^*(x) = b^*(x),$$

$$\langle x + (\log \circ b(x))^*, b(x) \rangle^{-1} = \langle x - (\log \circ a(x))^*, a^{(-1)}(x) \rangle.$$

Denote

$$[x^n] a^{(\ln m)}(x) = a_n^m, \quad [x^n] (x - (\log \circ a(x))^*) \circ a^{(\ln m)}(x) = c_n^m,$$

$$a_m(x) = \sum_{n=1}^{\infty} a_n^{mn} x^n, \quad c_m(x) = \sum_{n=1}^{\infty} c_n^{mn} x^n.$$

Construct the matrices A, C :

$$[\uparrow, 0]A = \varphi, \quad [\uparrow, n]A = a_n(x^n), \quad [\uparrow, 0]C = \varphi, \quad [\uparrow, n]C = c_n(x^n),$$

where φ is a some number,

$$A = \begin{pmatrix} \varphi & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_1^1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_2^2 & a_1^2 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_3^3 & 0 & a_1^3 & 0 & 0 & 0 & \dots \\ 0 & a_4^4 & a_2^4 & 0 & a_1^4 & 0 & 0 & \dots \\ 0 & a_5^5 & 0 & 0 & 0 & a_1^5 & 0 & \dots \\ 0 & a_6^6 & a_3^6 & a_2^6 & 0 & 0 & a_1^6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad C = \begin{pmatrix} \varphi & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & c_1^1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & c_2^2 & c_1^2 & 0 & 0 & 0 & 0 & \dots \\ 0 & c_3^3 & 0 & c_1^3 & 0 & 0 & 0 & \dots \\ 0 & c_4^4 & c_2^4 & 0 & c_1^4 & 0 & 0 & \dots \\ 0 & c_5^5 & 0 & 0 & 0 & c_1^5 & 0 & \dots \\ 0 & c_6^6 & c_3^6 & c_2^6 & 0 & 0 & c_1^6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is obvious that ($n > 0$)

$$[n, \rightarrow]A = [n, \rightarrow]\langle a^{(\ln n)}(x), x \rangle,$$

$$[n, \rightarrow]C = [n, \rightarrow]\langle x - (\log \circ a(x))^* \circ a^{(\ln n)}(x), x \rangle.$$

Since

$$(x - a^*(x) \circ a^{(-1)}(x)) \circ a^{(\ln m)}(x) = a^{(\ln m)}(x) - \frac{1}{\ln m} (a^{(\ln m)}(x))^*,$$

or

$$[x^n](x - (\log \circ a(x))^* \circ a^{(\ln m)}(x)) = \frac{\ln(m/n)}{\ln m} [x^n] a^{(\ln m)}(x),$$

then

$$\begin{aligned} [x^{nm}] Ax^m \circ (x - (\log \circ a(x))^* \circ a^{(-\ln m)}(x)) &= [x^{nm}] Cx^m \circ a^{(-\ln m)}(x) = \\ &= [x^n](x - (\log \circ a(x))^* \circ a^{(\ln n)}(x)) = 1, \quad n = 1; \quad = 0, \quad n > 1. \end{aligned}$$

Thus, up to the element equal φ ,

$$A = \langle x + (\log \circ b(x))^*, b(x) \rangle, \quad C = \langle x, b(x) \rangle,$$

$$[x^n] b^{(\ln m)}(x) = c_n^{mn} = \frac{\ln m}{\ln mn} [x^n] a^{(\ln mn)}(x).$$

Denote

$$\langle x, a^{(-\beta)}(x) \rangle^{-1} = \langle x, {}_{(\beta)}a^{(\beta)}(x) \rangle.$$

Then

$$[x^n] {}_{(\beta)}a^{(\beta \ln m)}(x) = \frac{\beta \ln m}{\beta \ln m + \beta \ln n} [x^n] a^{(\beta \ln m + \beta \ln n)}(x).$$

Denote

$$[n, \rightarrow]\langle x | \log \circ {}_{(\beta)}a(x) \rangle_{e^x} = {}_{(\beta)}\tilde{s}_n(x), \quad {}_{(0)}\tilde{s}_n(x) = \tilde{s}_n(x).$$

Then

$$\begin{aligned} {}_{(\beta)}a^{(\varphi)}(x) &= \sum_{n=1}^{\infty} \frac{\varphi}{\varphi + \beta \ln n} \frac{\tilde{s}_n(\varphi + \beta \ln n)}{n!} x^n, \\ {}_{(\beta)}\tilde{s}_n(x) &= x(x + \beta \ln n)^{-1} \tilde{s}_n(x + \beta \ln n). \end{aligned}$$

Example.

$$[x^n] \varepsilon^{(\varphi)}(x) = \frac{\varphi^{s(n)}}{f(n)}, \quad [x^n] {}_{(1)}\varepsilon^{(\varphi)}(x) = \frac{\varphi(\varphi + \ln n)^{s(n)-1}}{f(n)},$$

$s(1) = 0, \quad f(1) = 1, \quad s(n) = m_1 + m_2 + \dots + m_r, \quad f(n) = m_1!m_2!\dots m_r!,$
 $n = p_1^{m_1}p_2^{m_2}\dots p_r^{m_r}$ is the canonical decomposition of number n . From

$$({}_1\varepsilon^{(\varphi+\beta)}(x) = {}_1\varepsilon^{(\varphi)}(x) \circ {}_1\varepsilon^{(\beta)}(x))$$

we obtain analog of the Abel's generalized binomial formula:

$$(\varphi + \beta)(\varphi + \beta + \ln n)^{s(n)-1} = \sum_{d|n} \binom{n}{d}_f \varphi(\varphi + \ln d)^{s(d)-1} \beta(\beta + \ln(n/d))^{s(n/d)-1},$$

where

$$\binom{n}{d}_f = \frac{f(n)}{f(d)f(n/d)};$$

or, since

$$[x^n](x + (\log \circ {}_1\varepsilon(x))^*) \circ {}_1\varepsilon^{(\varphi)}(x) = [x^n]\varepsilon^{(\varphi+\ln n)}(x),$$

then

$$(\varphi + \beta + \ln n)^{s(n)} = \sum_{d|n} \binom{n}{d}_f (\varphi + \ln d)^{s(d)} \beta(\beta + \ln(n/d))^{s(n/d)-1}.$$

Since

$$({}_1\varepsilon^{(\varphi)}(x) = \langle x, {}_1\varepsilon(x) \rangle \varepsilon^{(\varphi)}(x), \quad \varepsilon^{(\varphi)}(x) = \langle x, \varepsilon^{(-1)}(x) \rangle {}_1\varepsilon^{(\varphi)}(x),$$

then

$$\varphi(\varphi + \ln n)^{s(n)-1} = \sum_{d|n} \binom{n}{d}_f \varphi^{s(d)} \ln d (\ln n)^{s(n/d)-1},$$

$$\varphi^{s(n)} = \sum_{d|n} \binom{n}{d}_f \varphi(\varphi + \ln d)^{s(d)-1} (\ln(1/d))^{s(n/d)}.$$

When $n = p^m$ formulas take the form of Abel's identities [9; p. 92-99]:

$$(\varphi + \beta)(\varphi + \beta + ma)^{m-1} = \sum_{k=0}^m \binom{m}{k} \varphi(\varphi + ka)^{k-1} \beta(\beta + (m-k)a)^{m-k-1},$$

$$(\varphi + \beta + ma)^m = \sum_{k=0}^m \binom{m}{k} (\varphi + ka)^k \beta(\beta + (m-k)a)^{m-k-1},$$

$$\varphi(\varphi + ma)^{m-1} = \sum_{k=0}^m \binom{m}{k} \varphi^k ka (ma)^{m-k-1},$$

$$\varphi^m = \sum_{k=0}^m \binom{m}{k} \varphi(\varphi + ka)^{k-1} (-ka)^{m-k}, \quad a = \ln p.$$

We generalize this example. Let $s_n(x)$ is the certain binomial sequence. Construct the series $a(x)$:

$$[x^n] a^{(\varphi)}(x) = \frac{u_n(\varphi)}{f(n)}, \quad u_n(x) = s_{m_1}(x) s_{m_2}(x) \dots s_{m_r}(x), \quad n = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r}.$$

Then

$$[x^n] {}_{(\beta)}a^{(\varphi)}(x) = \frac{\varphi}{\varphi + \beta \ln n} \frac{u_n(\varphi + \beta \ln n)}{f(n)},$$

$${}_{(\beta)}a^{(\varphi)}(x) = \langle x, {}_{(\beta)}a^{(\beta)}(x) \rangle a^{(\varphi)}(x), \quad a^{(\varphi)}(x) = \langle x, a^{(-\beta)}(x) \rangle {}_{(\beta)}a^{(\varphi)}(x),$$

$$\frac{\varphi}{\varphi + \beta \ln n} u_n(\varphi + \beta \ln n) = \sum_{d|n} \binom{n}{d}_f u_d(\varphi) \frac{\ln d}{\ln n} u_{n/d}(\beta \ln n),$$

$$u_n(\varphi) = \sum_{d|n} \binom{n}{d}_f \frac{\varphi}{\varphi + \beta \ln d} u_d(\varphi + \beta \ln d) u_{n/d}(\beta \ln(1/d)).$$

Since $u_{p^m}(x) = s_m(x)$, when $n = p^m$ formulas take the form of mutually inverse relations for the Lagrange series:

$$\frac{\varphi}{\varphi + ma} s_m(\varphi + ma) = \sum_{k=0}^m \binom{m}{k} s_k(\varphi) \frac{k}{m} s_{m-k}(ma),$$

$$s_m(\varphi) = \sum_{k=0}^m \binom{m}{k} \frac{\varphi}{\varphi + ka} s_k(\varphi + ka) s_{m-k}(-ka), \quad a = \beta \ln p.$$

Note the identities for the coefficients $\binom{n}{d}_f$, similar to the identities

$$\sum_{k=0}^n \binom{n}{k} = 2^n; \quad \sum_{k=0}^n \binom{n}{k} k (-1)^{n-k} = 0, \quad n \neq 1.$$

Since $\varepsilon(x) \circ \varepsilon(x) = \varepsilon^{(2)}(x)$, $\varepsilon^*(x) \circ \varepsilon^{(-1)}(x) = (\log \circ \varepsilon(x))^*$, then

$$\sum_{d|n} \binom{n}{d}_f = 2^{s(n)}; \quad \sum_{d|n} \binom{n}{d}_f \ln d (-1)^{s(n/d)} = 0, \quad n \neq p.$$

Generalization of the theorem 2 is the formula

$$b(x) = (x - (\log \circ a(x))^*) \circ \sum_{n=1}^{\infty} a^{(-\ln n)}(x^n) [x^n] b(x) \circ a^{(\ln n)}(x),$$

which follows from

$$[n, \rightarrow] \langle x - (\log \circ a(x))^*, a^{(-1)}(x) \rangle^{-1} = [n, \rightarrow] \langle a^{(\ln n)}(x), x \rangle.$$

References

- [1] L. Shapiro, S. Getu, W. Woan, L. Woodson, The Riordan group, Discrete Appl. Math., 34 (1991) 229-339.
- [2] R. Sprugnoli, Riordan arrays and combinatorial sums, Discrete Math., 132 (1994) 267-290.
- [3] W. Wang, T. Wang, Generalized Riordan arrays, Discrete Math., 308 (2008) 6466-6500.
- [4] Tian-Xiao He, Leetsch C. Hsu, Peter J.S. Shiue, The Sheffer Group and the Riordan Group, Discrete Appl. Math., 155 (2007) 1895-1909.
- [5] G. H. Hardy, Divergent Series, Oxford: Clarendon Press, 1949.
- [6] V. E. Hoggatt, Jr. and Paul S. Bruckman, H-convolution transform, The Fibonacci Quarterly, Vol. 13, № 4, 1975, 357-368.
- [7] E. V. Burlachenko, Riordan arrays and generalized Lagrange series, Mathematical Notes, Vol. 100, № 4, 2016, 531-539.

[8] R. L. Graham, D. E. Knuth and O. Patashnik, Concrete Mathematics, Addison-Wesley, 1989.

[9] J. Riordan, Combinatorial Identities, New York: Wiley, 1968.

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